Constructing Graceful Graphs by Extending Paths from Graceful Graphs

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Abstract

A graph with n vertices and m edges permits a graceful labeling if we can label its vertices with n distinct integers from 0 to m such that the absolute differences of adjacent vertices induce the edge labels from 1 to m. A graph which permits at least one such labeling is graceful. We investigate the construction of a graceful graph G' from a graceful graph G by considering the extension of a path from any vertex of G. Namely, we show that if a vertex in G has label x, $0 \leq x \leq m$, then extending a path of order greater than or equal to $2x(2x+1)$ from that vertex yields a graceful graph G'. This is a generalization of the ability to extend a path of any order from the vertex labeled 0 to construct another graceful graph.

1 Introduction

1.1 Definitions

Let G be a graph with a set of vertices $V(G)$ and a set of edges $E(G)$. Let m denote the size, $|E(G)|$, of G and n denote the order, $|V(G)|$, of G. Then a **graceful labeling** of G is an injective map $f : V(G) \to \{0, 1, ..., m\}$, such that the induced edge labeling g defined by,

$$g : E(G) \to \mathbb{N}, (u, v) \to |f(u) - f(v)|$$

produces a labeling of $E(G)$ which is a bijective map of the set $E(G)$ onto the set $\{1, 2, ..., m\}$. That is, the set of absolute differences of adjacent vertex pairs must contain no duplicates, and must include every number from 1 to m. A graph G is said to be **graceful** if it permits a graceful labeling.

When the graph G is a tree, $m = n - 1$. Thus, a tree is graceful if and only if there exists a vertex labeling which uses each member of the set $\{0, 1, ..., n-1\}$ exactly once – the labeling is bijective – and produces an induced edge labeling where $E(G) \to \{1, 2, ..., n-1\}$ is also bijective. For example, Figures 1 and 2 depict graceful labelings of different classes of graphs. Note that if the graph is not a tree, then some vertex labels are not used.

Let G be a graceful graph with size m, and a graceful labeling f. Every graceful labeling of a graph G has a **complementary labeling** f^* which is defined for each vertex $v \in V(G)$ by

$$f^*(v) = m - f(v)$$

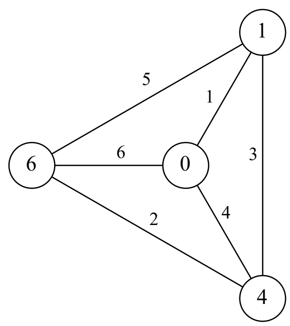

Figure 1: Graceful labeling of K_4

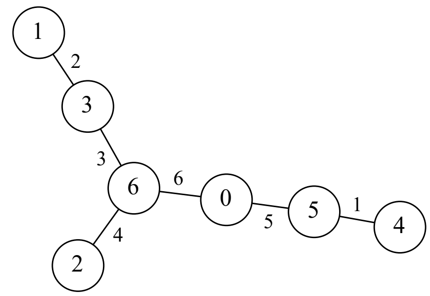

Figure 2: Graceful labeling of a tree

A complementary labeling of a graceful labeling is itself graceful. Note too that taking the complementary labeling of a graceful labeling "switches" the positions of the labels for 0 and m, and that it maintains the positions of the induced edge labels. For example, Figure 3 shows the complementary labeling of the graceful labeling of K_4 depicted in Figure 1. In addition, Figure 4 shows the complementary labeling of the graceful labeling of the tree depicted in Figure 2.

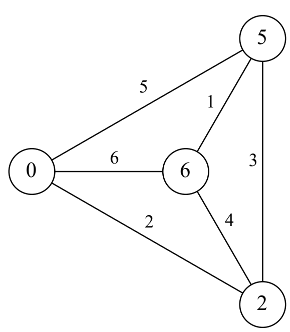

Figure 3: Complementary graceful labeling of K_4

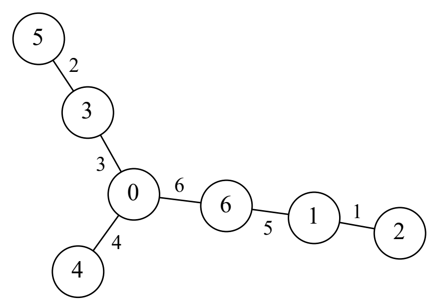

Figure 4: Complementary graceful labeling of a tree

1.2 Background

In 1967, Rosa introduced the idea of *valuations of a graph*, which he defined as an assignment of non-negative integer labels to the vertices of the graph with an induced edge labeling defined by the absolute differences of incident vertices. Rosa investigated four classes of valuations, one of which we now refer to as graceful labelings. Rosa used valuations to tackle the question of cyclic decomposition of complete graphs. Namely, he proved

Theorem 1 ([Ros67])**.** *Let G be a graph with m edges. If G is graceful, then the complete graph K_{2m+1} is G-decomposable.*

Although Rosa's result applies to all graphs, trees remain the most often studied. Historically, this is because Ringel and Kotzig originally conjectured that all complete graphs of order $2m+1$ are decomposable into any tree of order $m+1$. Hence, the infamous Ringel-Kotzig Conjecture, more commonly known as the Graceful Tree Conjecture:

Conjecture 1 (**Graceful Tree Conjecture**)**.** *All trees are graceful.*

Were the Graceful Tree Conjecture to become a theorem, then it would immediately follow that all complete graphs K_{2m+1} are decomposable into copies of any tree with m edges (or $m+1$ vertices). As such, this conjecture has inspired hundreds of authors to investigate graceful trees, which we also pursue here. Most authors have isolated (or defined) a class of trees and proven that that class is graceful. Some of the most famous such classes include caterpillars, symmetrical trees, some spiders, some lobsters, trees on fewer than 36 vertices, and a number of other particular cases ([Rob11], [Gal98]). In addition, some authors have considered the adjoinment of two graceful trees to form other graceful trees. We investigate an approach closer to the second, and consider the extension of paths from an arbitrary vertex on an arbitrary graceful graph.

2 Results

2.1 Terminology

First, some notes on terminology. Let v_0 be a vertex in $V(G_0)$ and v_1 be a vertex in $V(G_1)$. We use the phrase *pin vertex* v_1 *to vertex* v_0 to denote the unificiation of graphs G_0 and G_1 by equating vertices v_0 and v_1 as a single vertex v, as in Figure 5:

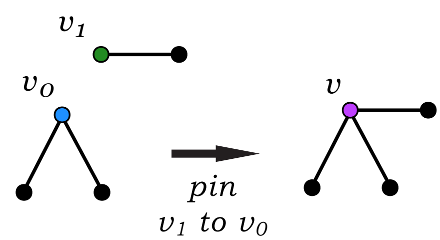

Figure 5: Example of pinning vertices

Second, a **caterpillar** is a tree such that when all of its leaves are removed, a path remains. Equivalently, a caterpillar can be thought of as an ordered series of stars (each with 0 or more leaves) joined by a path which connects the central vertices of adjacent stars. We use the second definition, and refer to this path which connects the stars as the **base path** of the caterpillar, and the leaves of the base as the **end vertices** of the caterpillar. For example, Figure 6 depicts a caterpillar composed of 4 stars with 1, 0, 2, and 3 leaves, from left to right; its base is blue and its end vertices are green.

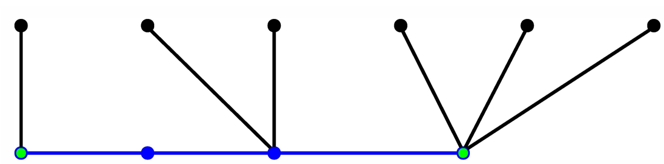

Figure 6: A caterpillar

2.2 Previous Results

We begin with a proof which, although not a novel result, demonstrates the methodology and reasoning for our later results.

Theorem 2. *Let G be a gracefully labeled graph, and let g_0 denote the vertex with label* 0*. Let C denote a caterpillar with k vertices in its base path and end vertices v_1 and v_k. Form G' by pinning vertex v_1 of C to vertex g_0 of G. Then G' has a graceful labeling which places the label* 0 *on v_k.*

Proof. We proceed by induction on the the number of vertices in the base of C, denoted k. Let v_k denote the k^{th} vertex in the base path, and l_k denote the number of leaves on each vertex v_k.

Base Case. Let G be a gracefully labeled graph with size m and a vertex g_0 labeled 0. Let C be a caterpillar whose base path has $k = 1$ vertices. Then C is a star, and pinning v_1 to g_0 is equivalent to extending l_k leaf vertices from g_0. A graceful labeling of the resulting graph G', with size m', can be generated by maintaining all of the labels in G, and placing the labels $\{m+1, ..., m+l_k\}$ on these l_k leaves of C. This induces all of the edge labelings $\{1, 2, ..., m + l_k = m'\}$ on G', so G' is graceful. Note that the label 0 is on $g_0 = v_1 = v_k$, the only base vertex of C.

Inductive Step. Let G be a gracefully labeled graph with size m and vertex g_0 labeled 0. Let C be a caterpillar whose base path has $k \geq 2$ vertices. Form G', with size m', by pinning v_1 of C to g_0. By the inductive assumption, there exists a labeling of G' up to and including v_{k-1} which uses the vertex labels $\{0, 1, ..., m' - l_k - 1\}$ and produces the induced edge labels $\{1, 2, ..., m' - l_k - 1\}$. Additionally, the inductive assumption places the label 0 on v_{k-1}. In order to produce a graceful labeling of G', we still need to place the labels $\{m' - l_k, ..., m'\}$. First, place the label $m' - l_k$ on v_k (see Figure 7). Then, take the complementary labeling of this partial labeling, in order to "swap" the labels on v_{k-1} and v_k (see Figure 8).

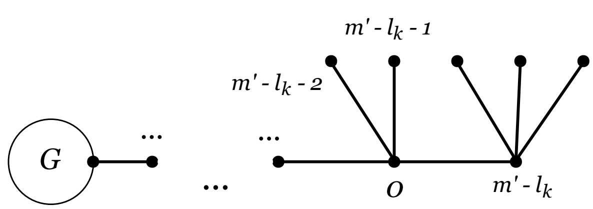

Figure 7: Partial graceful labeling

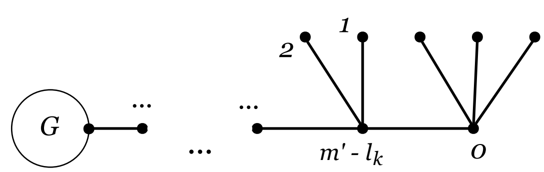

Figure 8: After complementary labeling

To complete the graceful labeling of G', place the labels $\{m'-l_k+1, m'-l_k+2, ..., m'\}$ on the leaves of v_k. Because v_k has label 0, this induces the edge labels $\{m'-l_k+1, m'-l_k+2, ..., m'\}$. This induces all of the edge labelings $\{0, 1, 2, ..., m + l_k + 1 = m'\}$, so G' is graceful. By the principle of mathematical induction, the result follows. *QED*

From Theorem 2, it naturally emerges that all caterpillars (and by extension, all paths) are graceful. This is not a novel result. Many have proven the gracefulness of caterpillars, including Rosa and Robeva ([Ros67], [Rob11]). However, this proof generalizes the result, such that any caterpillar C can be pinned to the 0 vertex of any gracefully labled graph G to produce another graceful graph.

The methodology of this proof exemplifies a crucial crux in the Graceful Tree Conjecture: Induction only works in specialized, somewhat contrived cases, and therefore often necessitates cumbersome terminology or specific classifications. In addition, such approaches place a special value on the vertex labeled 0, a phenomenon which is often studied for graceful graphs ([VB04], [Ros77], [Cav06]). These limitations account for why relatively little progress has been made on the general case of the conjecture, and also why it can be difficult to obtain results on even relatively simple classes of trees. Hence, we consider a technique for constructing graceful graphs from an arbitrary vertex on an arbitrary graceful graph.

First, note that because paths are a special case of caterpillars, Theorem 2 also applies to the extension of a path of any order from the vertex labeled 0:

Corollary 3. *Let G be a gracefully labeled graph, and let g_0 denote the vertex with label 0 in G. Form G' by extending a path P of any order from vertex g_0 of G. Then G' is graceful.*

2.3 Our Results

Now, we investigate a generalization of Corollary 3, and consider the extension of a path from any vertex of a graceful graph G.

Lemma 4. *Let G be a gracefully labeled graph with size m, and let g_x denote the vertex with label x, $0 \leq x \leq m$, in G. Let P denote a path of order $2x+1$. Form G' by adding an edge between a leaf vertex of P and vertex g_x of G. Then G' is graceful.*

Proof. To begin, add x to each vertex label in G, including g_x. This shifts the vertex labels of G up by x, without affecting the induced edge labels. Note that vertex g_x now holds label $x+x=2x$. Thus, vertex labels $\{0,1,...,x-1\}$ and $\{m+x+1,m+x+2,...,m+2x+1\}$ remain to be placed on path P, with necessary induced edge labels $\{m+1,m+2,...,m+2x+1\}$. By appropriately alternating between the high and low labels along the path, we can induce the correct edge labels while using all vertex labels:

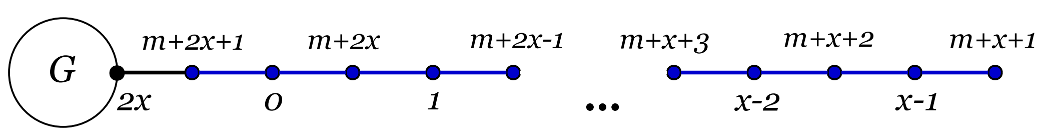

Figure 9: Labeling of path P – in blue – of order $2x+1$

However, to verify that this labeling can be placed on P for any value x, we must check that the number of large labels (placed above the path in Figure 9) exceed the number of small labels by 1. The set $\{m+x+1,m+x+2,...,m+2x+1\}$ has size $(m+2x+1)-(m+x+1)+1 = x+1$ and the set $\{0,1,...,x-1\}$ has size $(x-1)-(0)+1 = x$, as desired. As such, P permits the $2x+1$ labels, thereby producing a graceful labeling of G'. *QED*

Before proceeding to the next lemma, we begin with a remark on the previous. Consider the labeling in Figure 9, and denote this particular labeling by f. As before, denote the overall graph as G', which has size $m + 2x + 1$. Then, the complementary labeling f^* of f is defined as:

$$f^*(v) = (m + 2x + 1) - f(v), \quad \forall v \in V(G')$$

This complementary labeling is depicted in Figure 10.

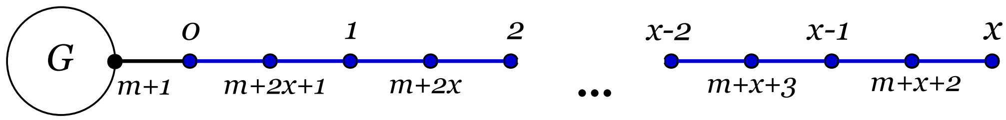

Figure 10: Complementary labeling of path P – in blue – of order $2x + 1$

Therefore, for a graph G' formed by extending a path of order $2x + 1$ from the vertex labeled x, there exists a graceful labeling of G' which places the label x on the leaf of this path. We use this remark later, but first consider the extension of a path of order $2x + 2$. Namely, the next result follows an argument almost identical to that of Lemma 4.

Lemma 5. *Let G be a gracefully labeled graph with size m, and let g_x denote the vertex with label x, $0 \leq x \leq m$, in G. Let P denote a path of order $2x + 2$. Form G' by adding an edge between a leaf vertex of P and vertex g_x of G. Then G' is graceful.*

Proof. To begin, add $x + 1$ to each vertex label in G, including g_x. This shifts the vertex labels of G up by $x + 1$, without affecting the induced edge labels. Thus, vertex labels $\{0, 1, ..., x\}$ and $\{m + x + 2, m + x + 3, ..., m + 2x + 2\}$ remain to be placed on path P, with necessary induced edge labels $\{m + 1, m + 2, ..., m + 2x + 2\}$. We again alternate between the high and low labels along the path:

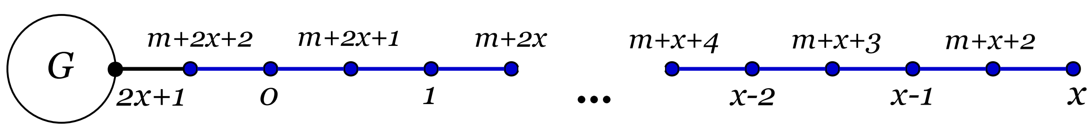

Figure 11: Labeling of path P – in blue – of order $2x + 2$

Once again, we must verify that this labeling can be placed on P for any value x. Namely, the number of large labels (placed above the path in Figure 11) must equal the number of small labels, which can be checked to be $x + 1$ for both sets. As such, P permits the total $2x + 2$ labels, thereby producing a graceful labeling of G'. *QED*

With these lemmas in place, we can prove a more general, more interesting result.

Theorem 6. *Let G be a gracefully labeled graph with size m, and let g_x denote the vertex with label x, $0 \leq x \leq m$, in $V(G)$. Let P denote a path of order n. Form G' by adding an edge between a leaf vertex of P and vertex g_x of G. If n is a non-negative integer linear combination of $2x + 1$ and $2x + 2$, then G' is graceful.*

Proof. First, consider the case when P is of order $n = 2x + 1$, for which there is a graceful labeling – depicted in Figure 10 – which places the label x on the leaf vertex at the end of

P. Similarly, for the case when P is of order $n = 2x + 2$, there exists a graceful labeling – depicted in Figure 11 – which places the label x on the leaf vertex at the end of P. Either way, Lemmas 4 and 5 allow for the label x to be placed at the end of the added path. Now, the general case for n.

Because n is a non-negative integer linear combination of $2x + 1$ and $2x + 2$, there exist $s, t \in \mathbb{Z}_{\geq 0}$ such that $n = s(2x + 1) + t(2x + 2)$. Thus, a graceful labeling of G' can be constructed by iteratively applying Lemma 4 – with complementary labeling – s times and Lemma 5 t times to the vertex with label x in G. This produces an extended path of order $s(2x + 1) + t(2x + 2)$ as desired, thereby constructing the graceful graph G'. *QED*

Framing Theorem 6 with an observation from number theory establishes a lower bound on the order of paths which maintain gracefulness when extended from an arbitrary vertex on a graceful graph Gx. First, the observation:

Lemma 7. *Let $a, b \in \mathbb{Z}_{\geq 0}$ be two relatively prime integers. Let n represent the smallest integer such that all integers $\geq n$ are a non-negative integer linear combination of a and b. Then $n = (a - 1)(b - 1)$.*

Proof. To prove that $n = (a - 1)(b - 1)$, we show that $n \leq (a - 1)(b - 1)$ and $n \geq (a - 1)(b - 1)$. Without loss of generality, assume that $a < b$. Let S be the set of all non-negative integer linear combinations of a and b. Because a and b are relatively prime, the set $R = \{0, b, 2b, ..., (a - 1)b\} \subset S$ contains a unique representative of each residue class mod a.

By adding non-negative multiples of a to a consecutive integers in S, we cover every subsequent integer as a linear combination of a and b. Thus, to show that $n \leq (a-1)(b-1)$, it suffices to show that $(a - 1)b - i$ is in S for each integer i, $1 \leq i \leq a - 1$. For each such i, let $k_i b$ be the element of R with the same residue as $(a - 1)b - i$ modulo a. By construction, $a - 1 \geq k_i$ so $(a - 1)b \geq k_i b$. Using this, and the assumption that $a < b$, it follows that $(a - 1)b - i > k_i b$. However, $(a - 1)b - i$ and $k_i b$ are in the same residue class modulo a, so $(a - 1)b - i$ must be some positive multiple of a greater than $k_i b$. Therefore, $(a - 1)b - i = k_i b + \ell a$, for some positive integer ℓ, and is therefore in S. This yields a consecutive integers in S, which begin at $(a - 1)(b - 1)$ and end at $(a - 1)b$.

To show that $n \geq (a - 1)(b - 1)$, we show that R is the set of smallest representatives modulo a among all elements of S. That is, $zb - a$ is not in S, for any integer z, $1 \leq z < a$. Because a and b are relatively prime, zb is not divisible by a. As such, $zb - a$ cannot be in S, as it is not reducible to a multiple of the smaller number a. Therefore, R is the set of smallest representatives and $n \leq (a - 1)(b - 1)$. The lemma follows. *QED*

Taking the relatively prime integers a and b in Lemma 7 to be $2x + 1$ and $2x + 2$ as in Theorem 6:

Corollary 8. *Let G be a gracefully labeled graph with size m, and let g_x denote the vertex with label x, $0 \leq x \leq m$, in $V(G)$. Let P denote a path of order n. Form G' by adding an edge between a leaf vertex of P and vertex g_x of G. If $n \geq 2x(2x + 1)$, then G' is graceful.*

Applying Theorem 6 to the case when $x = 0$ demonstrates that it is a generalization of the result found in Corollary 3: Extending a path of any order from the vertex labeled 0 for some graceful labeling of a graph G produces another graceful graph G'. However, this theorem allows for other cases. For example, considering the case of $x = 1$ exemplifies a more unique application of Theorem 6, and is stated in the next corollary.

Corollary 9. *Let G be a gracefully labeled graph. Let g_1 denote the vertex with label 1. Let P denote a path of order n where $n = 3$, $n = 4$, or $n \geq 6$. Form G' by adding an edge between a leaf vertex of P and vertex g_1 of G. Then G' is graceful.*

3 Acknowledgements

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